Introduction to Mathematical Quantum Theory

Text of the Exercises

– 12.05.2020 –

Teacher: Prof. Chiara Saffirio

Assistent: Dr. Daniele Dimonte – daniele.dimonte@unibas.ch

Exercise 1

Let $\mathcal{H} := L^2(\mathbb{R})$ and $P := -i\partial_x$ the momentum operator defined on the domain $\mathcal{D}(P) := H^1(\mathbb{R})$ as $P\psi(x) = -i\frac{\partial\psi}{\partial x}(x)$. Consider for any $\lambda \in \mathbb{R}$ the bounded operator T_λ defined for any $\psi \in \mathcal{H}$ as $T_\lambda\psi(x) = \psi(x-\lambda)$.

Prove that $\{T_\lambda\}_{\lambda\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group and that

$$T_\lambda = e^{i\lambda P} = e^{\lambda\partial_x}. \tag{1}$$

Exercise 2

Let $\mathcal{H}$ be an Hilbert space, A a symmetric operator and $\mu > 0$ a positive real number. Prove that the following are equivalent.

a A is self-adjoint.

b $\mathrm{Ran}(A + i\mu\,\mathrm{id}) = \mathrm{Ran}(A - i\mu\,\mathrm{id}) = \mathcal{H}$.

Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Let $U \in \mathcal{B}(\mathcal{H})$. Prove that U is unitary if and only if there exist a self-adjoint operator A on $\mathcal{H}$ such that $U = e^{iA}$.

Exercise 4

Let $\mathcal{H}$ be an Hilbert space and $A_+,\ A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^*\right] = \mathrm{id}, \tag{2}$$
$$\left[A_+, A_-\right] = \left[A_+, A_-^*\right] = 0. \tag{3}$$

Let moreover $\eta,\ \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta\left(A_+^*A_+ + A_-^*A_-\right) + \zeta\left(A_+^*A_-^* + A_+A_-\right). \tag{4}$$

a Prove that H is self-adjoint.

b Prove that there exist operators $C_\pm$ and numbers $\alpha,\ \beta \in \mathbb{R}$ such that

$$\left[C_\pm, C_\pm^*\right] = \mathrm{id}, \tag{5}$$
$$\left[C_+, C_-\right] = \left[C_+, C_-^*\right] = 0, \tag{6}$$
$$H = \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta. \tag{7}$$

Hint: Define

$$C_\pm := \gamma_\pm A_\pm + \xi_\pm A_\mp^* \tag{8}$$

for some $\gamma_\pm,\ \xi_\pm \in \mathbb{R}$*. Use* (5) *and* (6) *to deduce that* $\gamma_+ = \gamma_-$*,* $\xi_+ = \xi_-$ *and that* $\gamma_\pm^2 - \xi_\pm^2 = 1$*. Calculate* $C_\pm^* C_\pm$ *and deduce* (7)*.*